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Canonical analysis of inhomogeneous Dark Energy Model and theory of limiting curvature

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Abstract: This paper is devoted to the canonical analysis of inhomogeneous Dark Energy Model and the model of limiting curvature that were proposed recently by Chamseddine and V. Mukhanov. We argue these models are well defined and have similar properties as a system consisting from general gravity action and action for incoherent dust.

Keywords: Classical Theories of Gravity, Models of Quantum Gravity

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Contents

1 Introduction

Few years ago a new interesting model of mimetic dark matter was suggested in [1] and was further elaborated in [2, 3].[1] In mimetic gravity it is possible to describe the dark components of the Universe as a purely geometrical effect, without the need of introducing additional matter fields. This description can be achieved using very simple but remarkable idea. The physical metric $g_{\mu\nu}^{\text{phys}}$ is considered to be a function of a scalar field ϕ and a fundamental metric $g_{\mu\nu}$, where the physical metric is defined as[2]

$$g_{\mu\nu}^{\text{phys}} = \left(-g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right) g_{\mu\nu} \ . \tag{1.1}$$

The physical metric $g_{\mu\nu}^{\text{phys}}$ is invariant with respect to the Weyl transformation of the metric $g_{\mu\nu}$,

$$g'_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x) \ . \tag{1.2}$$

Then it was shown in [1] and in [2] that the ordinary Einstein-Hilbert action constructed using the physical metric $g_{\mu\nu}^{\text{phys}}$ possesses many interesting properties since this model can be interpreted as a conformal extension of Einstein's general theory of relativity. The local Weyl invariance is ensured by introducing an extra degree of freedom that as was shown in [1] has the form of pressureless perfect fluid that, according to [1], can mimic the behavior of a real cold dark matter.

This mimetic dark matter proposal [1] was recently generalized to so called "inhomogeneous dark energy model" in [5]. In this paper the original mimetic model was extended in

[1]For review and extensive list of references, see [4].

[2]We follow the convention used in [2] and we also consider the space-time metric of the signature $(-,+,+,+)$.

oder to describe arbitrary inhomogeneous dark energy in any scale and that can contribute to the gravitational instability at late time. This theory is described by the action[3]

$$S=\frac{1}{2}\int d^4x\sqrt{-g}\left[R(g_{\mu\nu})-\lambda\left(1+g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right)-\lambda_a g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi-V(\phi^a)\right]\,,\tag{1.3}$$

where $a=1,\ldots,D$ where D counts the number of the scalar fields ϕ^a. Generally we can have D arbitrary but the simplest possibility is to take $a=1$ while the most convenient case is to consider $D=3$ [5] since then we can identify the scalars with the synchronous coordinates. The variation of the action with respect to Lagrange multipliers λ and λ_a gives following equations of motion

$$\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x^\nu}g^{\mu\nu}=-1\,,\qquad \frac{\partial\phi^a}{\partial x^\mu}\frac{\partial\phi}{\partial x^\nu}g^{\mu\nu}=0\,.\tag{1.4}$$

It was shown in [5] that at synchronous coordinate system where $g_{00}=-1\,,g_{0i}=0\,,i=1,2,3$ so that

$$ds^2=-dt^2+h_{ij}dx^idx^j\tag{1.5}$$

the general solution of these equations in this coordinate system are

$$\phi=t\,,\qquad \phi^a=\phi^a(x^i)\tag{1.6}$$

so that the fields ϕ^a are time independent functions and the potential can be time independent function of the spatial coordinates and it leads to cosmological-like constant in the Einstein equations. It was also argued here that the inhomogeneities in the distribution of dark energy can have an impact on the power spectrum at large scales and the formation of structure of the universe.

All these facts makes the proposal [5] very interesting and deserves further study. Certainly it would be very useful to find Hamiltonian formulation of this theory. In fact, canonical analysis of the original mimetic dark matter model was very carefully performed in [6][4] where we identified all constraints and determined number of physical degrees of freedom. We also shown that by solving the second class constraints the resulting Hamiltonian for the scalar field is linear in the momentum conjugate to the scalar field. The presence of the linear momentum signals instability of the theory since the Hamiltonian is not bounded from below for certain type of initial configurations and consequently it can become unstable.

There is a natural question how the situation changes when we consider the model presented in [5] and this is one of the goals of the present paper. It turns out that the Hamiltonian analysis is rather non-trivial and depends on the number of additional scalar fields ϕ^a. In case when $a\neq 1$ we find new primary constraints that have to be taken into account. Then we also derive corresponding secondary constraints. Despite of the complexity of the Hamiltonian analysis we find that after solving second class constraints the Hamiltonian constraint is linear in momentum conjugate to the scalar field ϕ. On the

[3]The inhomogeneous extension of mimetic gravity is based on Einstein frame version of this theory where the manifest Weyl invariance (1.2) is lost. For more details, see for example [6].

[4]See also [7].

other hand we show that this contribution can be rewritten to the form corresponding to the Hamiltonian constraint for the dust [13] which is well defined system with quadratic Hamiltonian bounded from below. This analysis shows that the mimetic model and model studied here could be stable.

The final part of our paper is devoted to the Hamiltonian analysis of the model that was introduced very recently in [8, 9] which is very interesting proposal how to resolve singularities in general relativity. It is based on the idea of modification of classical general relativity at high curvatures by incorporating limiting curvature. If this limiting curvature is few order of magnitude below the Planckian value we can ignore quantum gravity effects. This remarkable proposal is again based on the existence of the scalar field that obeys the equation

$$g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = -1 \ . \tag{1.7}$$

It was also presumed that the theory is invariant under shift symmetry $\phi \to \phi + \text{const.}$ which implies that the potential $V(\phi)$ is absent. On the other hand a new additional term $f(\Box\phi)$ was added to the action, where the function f was suggested to have the form

$$f(\Box\phi) = 1 - \sqrt{1 - \frac{(\Box\phi)^2}{\epsilon_m}} + \dots \ . \tag{1.8}$$

It was shown in [8, 9] that this theory resolves singularities in Friedmann and Kasner universes. In other words the contracting universes bounce at the limiting curvature and all curvature invariants are regular and bounded by the values characterized by ϵ_m. Then it was shown in [9] that the physical singularity of the Schwarzschild black hole can be removed as well.

Since this proposal is based on an extension of mimetic action with a new term that depends on $\Box\phi$ we should study whether these higher derivatives do not imply instability of this theory. In order to answer this question we perform Hamiltonian analysis of this theory. By introducing two auxiliary fields we rewrite the action to the form that contains the first order derivatives only and hence it is suitable for the Hamiltonian analysis. We determine corresponding Hamiltonian and we also find structure of the constraints. Surprisingly we find that the presence of the higher derivative term does not lead to the existence of additional degree of freedom. This fact is a consequence of the presence of the constraint (1.7) in the action which implies that this theory is degenerate while Ostrogradsky's theorem is strictly speaking valid for non-degenerate theories only.[5]

In conclusion, we mean that the proposal suggested in [8, 9] is very remarkable and deserves further study. In particular, these theories could be useful for "deparameterising of the theory of gravity" [12, 13]. Briefly, this idea is based on a presumption that the Hamiltonian constraint can be written in the form $\mathcal{H}(\mathbf{x}) = \pi(\mathbf{x}) + \mathcal{K}(\mathbf{x})$, where $\pi(\mathbf{x})$ is the momentum conjugate to the scalar field $\phi(\mathbf{x})$ and where $\mathcal{K}$ is positive function on phase space which depends neither on ϕ or π. Then it is possible to construct physical observable and the function $\mathbf{K} = \int d^3\mathbf{x}\mathcal{K}(\mathbf{x})$ is the natural physical Hamiltonian that generates the time evolution of the observables, see [12]. It would be very interesting to see whether

[5]For recent review of Ostrogradsky's theorem, see [15].

models presented in [5, 8, 9] could allow such a construction. We currently analyze this problem.

The structure of this paper is as follows. In the next section 2 we perform Hamiltonian analysis of Inhomogeneous Dark Energy model as was formulated in [5]. Then in section 3 we consider simpler formulation of this model when the number of additional scalar fields is equal to one. In section 4 we perform Hamiltonian analysis of the model [8, 9]. Finally in conclusion 5 we outline our results and suggest possible extension of this work.

2 Hamiltonian analysis of inhomogeneous Dar Energy Model

In this section we perform Hamiltonian analysis of the action (1.3) in the full generality when we will presume that $a = 1, \dots, D$. In order to find its Hamiltonian form we use the following $3+1$ decomposition of the metric $g_{\mu\nu}$ [10, 11]

$$\begin{aligned} g_{00} &= -N^2 + N_i h^{ij} N_j , & g_{0i} &= N_i , & g_{ij} &= h_{ij} , \\ g^{00} &= -\frac{1}{N^2} , & g^{0i} &= \frac{N^i}{N^2} , & g^{ij} &= h^{ij} - \frac{N^i N^j}{N^2} , \end{aligned} \tag{2.1}$$

where we have defined h^{ij} as the inverse to the induced metric h_{ij} on the Cauchy surface Σ_t at each time t,

$$h_{ik} h^{kj} = \delta_i^{\ j} , \tag{2.2}$$

and we denote $N^i = h^{ij} N_j$. The four dimensional scalar curvature in $3+1$ formalism has the form

$$R(g_{\mu\nu}) = K_{ij} \mathcal{G}^{ijkl} K_{kl} + R + \frac{2}{\sqrt{-g}} \partial_\mu (\sqrt{-g} n^\mu K) - \frac{2}{\sqrt{h} N} \partial_i (\sqrt{h} h^{ij} \partial_j N) , \tag{2.3}$$

where the extrinsic curvature of the spatial hypersurface Σ_t at time t is defined as

$$K_{ij} = \frac{1}{2N} \left(\frac{\partial h_{ij}}{\partial t} - D_i N_j - D_j N_i \right) , \tag{2.4}$$

with D_i being the covariant derivative determined by the metric h_{ij}, and where the de Witt metric is defined as

$$\mathcal{G}^{ijkl} = \frac{1}{2} (h^{ik} h^{jl} + h^{il} h^{jk}) - h^{ij} h^{kl} \tag{2.5}$$

with inverse

$$\mathcal{G}_{ijkl} = \frac{1}{2} (h_{ik} h_{jl} + h_{il} h_{jk}) - \frac{1}{2} h_{ij} h_{kl} \tag{2.6}$$

that obeys the relation

$$\mathcal{G}_{ijkl} \mathcal{G}^{klmn} = \frac{1}{2} (\delta_i^m \delta_j^n + \delta_i^n \delta_j^m) . \tag{2.7}$$

Further, n^μ is the future-pointing unit normal vector to the hypersurface Σ_t, which is written in terms of the ADM variables as

$$n^0 = \sqrt{-g^{00}} = \frac{1}{N} , \qquad n^i = -\frac{g^{0i}}{\sqrt{-g^{00}}} = -\frac{N^i}{N} . \tag{2.8}$$

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Note that we are not interested in the boundary terms so that the terms proportional to total derivatives are not important for us and can be ignored.

Now inserting this $3+1$ decomposition into the action (1.3) we obtain

$$
\begin{aligned}
S &= \frac{1}{2}\int dt d^3\mathbf{x}\sqrt{h}N[K_{ij}\mathcal{G}^{ijkl}K_{kl} + R + \lambda(-1 + \nabla_n\phi\nabla_n\phi - h^{ij}\partial_i\phi\partial_j\phi) \\
&\quad + \lambda_a\nabla_n\phi^a\nabla_n\phi - \lambda_a h^{ij}\partial_i\phi^a\partial_j\phi - V(\phi^a)] \,,
\end{aligned} \tag{2.9}
$$

where

$$
\nabla_n\phi = \frac{1}{N}(\partial_t\phi - N^i\partial_i\phi) \,, \tag{2.10}
$$

and where we ignored boundary terms. Before we proceed further we should also stress one important point. Since $V(\phi^a)$ is a scalar function of ϕ^a the only possibility how to construct scalar from the vectors ϕ^a is to perform contractions of these two vectors. In order to do this we have to introduce general metric ω_{ab} on the space spanned by ϕ^a. We will presume that ω_{ab} is constant with inverse ω^{ab}. Of course, the simplest possibility is $\omega_{ab} = \delta_{ab}$ but we will keep ω_{ab} general. Then $\lambda^a = \omega^{ab}\lambda_b$.

Now we can easily derive the momenta conjugate to $h_{ij}, \Phi, \lambda, \lambda_a$ and ϕ, ϕ^a from the action (2.9) as

$$
\begin{aligned}
\pi^{ij} &= \frac{\delta L}{\delta\partial_t h^{ij}} = \frac{1}{2}\sqrt{h}\mathcal{G}^{ijkl}K_{kl}, \qquad \pi_N = \frac{\delta L}{\delta\partial_t N} \approx 0 \,, \qquad \pi_i = \frac{\delta L}{\delta\partial_t N^i} \approx 0 \,, \\
p_\phi &= \frac{\delta L}{\delta\partial_t\phi} = \lambda\sqrt{h}\nabla_n\phi + \frac{1}{2}\sqrt{h}\lambda_a\nabla_n\phi^a \,, \\
k^a &= \frac{\delta L}{\delta\partial_t\lambda_a} \approx 0 \,, \qquad k = \frac{\delta L}{\delta\partial_t\lambda} \approx 0 \,, \\
p_a &= \frac{\delta L}{\delta\partial_t\phi^a} = \frac{1}{2}\sqrt{h}\lambda_a\nabla_n\phi \,.
\end{aligned} \tag{2.11}
$$

Very interesting is the expression for p_a since it implies

$$
\frac{\lambda_b\lambda^a}{\lambda^c\lambda_c}p_a = p_b \tag{2.12}
$$

that leads to the following set of $D-1$ primary constraints

$$
\Sigma_a = P_a{}^b p_b \approx 0 \,, \qquad \lambda^a\Sigma_a = 0 \,, \tag{2.13}
$$

where $P_b{}^a = \delta_b^a + \frac{\lambda_b\lambda^a}{\lambda^c\lambda_c}$ is the projector to the space orthogonal to one dimensional space spanned by λ_a since

$$
P_b{}^a\lambda_a = 0 \,. \tag{2.14}
$$

Now it is easy to determine corresponding bare Hamiltonian

$$
\begin{aligned}
H &= \int d^3\mathbf{x}(\pi^{ij}\partial_t h_{ij} + p_\phi\partial_t\phi + p_a\partial_t\phi^a - \mathcal{L}) \\
&= \int d^3\mathbf{x}(N\mathcal{H}_T + N^i\mathcal{H}_i) \,,
\end{aligned} \tag{2.15}
$$

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where

$$\begin{aligned}
\mathcal{H}_T &= \frac{2}{\sqrt{h}}\pi^{ij}\mathcal{G}_{ijkl}\pi^{kl} - \sqrt{h}R + \frac{2}{\sqrt{h}\lambda_a\lambda^a}p_\phi(\lambda^c p_c) - \frac{2\lambda}{\sqrt{h}(\lambda^a\lambda_a)^2}(\lambda^b p_b)^2 \\
&\quad + \frac{1}{2}\lambda\sqrt{h}(1 + h^{ij}\partial_i\phi\partial_j\phi) + \frac{1}{2}\sqrt{h}\lambda_a h^{ij}\partial_i\phi^a\partial_j\phi + \frac{1}{2}\sqrt{h}V(\phi^a)\,, \\
\mathcal{H}_i &= -2h_{ik}D_j\pi^{kj} + p_\phi\partial_i\phi + p_a\partial_i\phi^a\ . \qquad (2.16)
\end{aligned}$$

Before we proceed further we should stress one important point which is related to the fact that k^a do not Poisson commute with projector $P_a{}^b$. Explicitly we find following Poisson brackets

$$\{k^a(\mathbf{x}), \Sigma_b(\mathbf{y})\} = \frac{\lambda^c p_c}{\lambda^d\lambda_d}P^a_b\delta(\mathbf{x}-\mathbf{y})\,, \qquad (2.17)$$

that however also implies that

$$\{\lambda_a k^a(\mathbf{x}), \Sigma_b(\mathbf{y})\} = 0\ . \qquad (2.18)$$

We see that it is natural to split k^a into $\tilde{k}^a \equiv k^a - \frac{\lambda^b k_b}{\lambda^d\lambda_d}\lambda^a = P^a_b k^b$ that is orthogonal to λ_a and their complement which is projection of k^a along λ_a defined as

$$\psi \equiv \lambda^a k_a\ . \qquad (2.19)$$

Note that $\tilde{k}^a$ has $D-1$ independent components since it obeys $\tilde{k}^a\lambda_a = 0$ by definition. It is important that there are non-zero Poisson brackets between $\tilde{k}^a$ and Σ_b equal to

$$\left\{\tilde{k}^a(\mathbf{x}), \Sigma_b(\mathbf{y})\right\} = \frac{\lambda^d p_d}{\lambda^e\lambda_e}P^a_b\delta(\mathbf{x}-\mathbf{y})\ . \qquad (2.20)$$

In other words $\tilde{k}^a$ and Σ_a are sets of $2(D-1)$-second class constraints. Then we introduce extended form of Hamiltonian with all primary constraints included

$$H_T = \int d^3\mathbf{x}(N\mathcal{H}_T + N^i\tilde{\mathcal{H}}_i + v^\psi\psi + v^k k + v^N\pi_N + v^i\pi_i + w^a\Sigma_a + v_a\tilde{k}^a)\,, \qquad (2.21)$$

where v^ψ, v^k, v^N, v^i are unspecified Lagrange multipliers corresponding to the primary constrains. On the other hand w^a, v_a belong to the space orthogonal to subspace generated by λ_a. Further, it is convenient to extended $\mathcal{H}_i$ in the following way

$$\tilde{\mathcal{H}}_i = \mathcal{H}_i + k\partial_i\lambda + k^a\partial_i\lambda_a\ . \qquad (2.22)$$

As the first step we analyze the requirement of preservation of the primary constraints during the time evolution of the system. In case of the constraints $\tilde{k}^a$ and Σ_a we obtain

$$\begin{aligned}
\partial_t\tilde{k}^a &= \left\{\tilde{k}^a, H_T\right\} = \int d^3\mathbf{x}\left(N\left\{\tilde{k}^a, \mathcal{H}_T(\mathbf{x})\right\} + w^b(\mathbf{x})\left\{\tilde{k}^a, \Sigma_b(\mathbf{x})\right\}\right) = 0\,, \\
\partial_t\Sigma_a &= \{\Sigma_a, H_T\} = \int d^3\mathbf{x}\left(N\{\Sigma_a, \mathcal{H}_T(\mathbf{x})\} + v_b(\mathbf{x})\left\{\Sigma_a, \tilde{k}^b(\mathbf{x})\right\}\right) = 0\ . \qquad (2.23)
\end{aligned}$$

Since w^b belong to the subspace transverse to λ_a we find that $w^b(\mathbf{y})\left\{\tilde{k}^a(\mathbf{x}),\Sigma_b(\mathbf{y})\right\} = \frac{\lambda^d p_d}{\lambda^e\lambda_e}P^a_b w^b\delta(\mathbf{x}-\mathbf{y}) = \frac{\lambda^d p_d}{\lambda^a\lambda_e}w^a\delta(\mathbf{x}-\mathbf{y})$ and hence the first equation in (2.23) has the solution

$$w^a = \frac{1}{2}\frac{\lambda^e\lambda_e}{\lambda^d p_d}NP^a_b\partial_j\phi^b h^{ij}\sqrt{h}\partial_j\phi \tag{2.24}$$

while the second one implies

$$v_a = \frac{1}{2}\frac{\lambda^e\lambda_e}{\lambda^d p_d}P_a{}^b\partial_j\lambda_b N\sqrt{h}h^{ij}\partial_j\phi \ . \tag{2.25}$$

Since $\tilde{k}^a, \Sigma_a$ have vanishing Poisson brackets with remaining primary constraints they effectively decouple.

Now we proceed to the analysis of the preservation of the primary constraints $\pi_N \approx 0, \pi_i \approx 0, k, \sigma \approx 0$. As usual the requirement of the preservation of the constraints π_N, π_i implies the secondary constraints

$$\mathcal{H}_T \approx 0\,, \qquad \tilde{\mathcal{H}}_i \approx 0\ . \tag{2.26}$$

For further analysis we introduce the smeared form of these constraints

$$\mathbf{T}_T(N) = \int d^3\mathbf{x}N\mathcal{H}_T, \qquad \mathbf{T}_S(N^i) = \int d^3\mathbf{x}N^i\tilde{\mathcal{H}}_i\ . \tag{2.27}$$

The requirement of the preservation of the constraint $k \approx 0$ implies

$$\partial_t k = \{k, H\} = N\left(\frac{2}{\sqrt{h}(\lambda^a\lambda_a)^2}(\lambda^b p_b)^2 - \frac{1}{2}\sqrt{h}(1 + h^{ij}\partial_i\phi\partial_j\phi)\right) \equiv N\Omega \approx 0\ . \tag{2.28}$$

Let us now proceed to the analysis of time evolution of the constraint ψ. Since ψ has zero Poisson bracket with Σ_a the requirement of its preservation during time evolution of the system implies new constraint. Explicitly, we find

$$\partial_t\psi = \{\psi, H_T\} = N\Sigma \approx 0\,, \tag{2.29}$$

where

$$\Sigma = \frac{2p_\phi(\lambda^a p_a)}{\sqrt{h}(\lambda^a\lambda_a)} - \frac{4\lambda(\lambda^b p_b)^2}{\sqrt{h}(\lambda^a\lambda_a)^2} - \frac{1}{2}\sqrt{h}h^{ij}\lambda_a\partial_i\phi^a\partial_j\phi \approx 0\ . \tag{2.30}$$

In summary we have following set of the second class constraints $\Psi_A = (\tilde{k}^a, \Sigma_a, k, \psi, \Omega, \Sigma)$. Now the matrix of Poisson brackets between these second class constraints has schematic form

$$\triangle_{AB} \equiv \{\Psi_A, \Psi_B\} = \begin{pmatrix} 0 & * & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & * & * & * & * \\ * & * & * & * & * & * \end{pmatrix}\,, \tag{2.31}$$

where $*$ means non-zero elements. Then it is easy to see that the inverse matrix has schematic form

$$\triangle^{AB} = \begin{pmatrix} * & * & * & * & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 \end{pmatrix} . \tag{2.32}$$

Finally we determine Poisson brackets between $\mathcal{H}_T$ and $\mathcal{H}_i$. We use their smeared form and we find

$$\begin{aligned} \{\mathbf{T}_T(N), \mathbf{T}_T(M)\} &= \mathbf{T}_S((N\partial_i M - M\partial_i N)h^{ij}) + \int d^3\mathbf{x}(N\partial_i M - M\partial_i N)h^{ij}\partial_j\phi^a\Sigma_a \,, \\ \left\{\mathbf{T}_S(N^i), \mathbf{T}_T(M)\right\} &= \mathbf{T}_T(N^i\partial_i M) \,, \\ \left\{\mathbf{T}_S(N^i), \mathbf{T}_S(M^j)\right\} &= \mathbf{T}_S(N^j\partial_j M^i - M^j\partial_j N^i) \,. \end{aligned} \tag{2.33}$$

Since all second class constraints are invariant under spatial diffeomorphism we find that they have weakly vanishing Poisson brackets with $\mathbf{T}_S(N^i)$. On the other hand it is easy to see that there are non-zero Poisson brackets between $\mathcal{H}_T$ and some of the constraints Ψ_A. Then it is convenient to introduce following constraint

$$\tilde{\mathcal{H}}_T = \mathcal{H}_T - \{\mathcal{H}_T, \Psi_A\}\triangle^{AB}\Psi_B \,, \tag{2.34}$$

where the sumation over A includes integration over space coordinates. From (2.34) we easily find that $\left\{\tilde{\mathcal{H}}_T, \Psi_B\right\} = 0$. This relation ensures that $\tilde{\mathcal{H}}_T$ is the first class constraint. In summary, we have four first class constraints $\tilde{\mathcal{H}}_T, \tilde{\mathcal{H}}_i$ as it is expected for the theory invariant under full diffeomorphism. Note also that $\tilde{\mathcal{H}}_T$ coincides with $\mathcal{H}_T$ when all second class constraints strongly vanish.

Finally we determine the structure of the Dirac brackets. Let us denote the Poisson bracket between all canonical variables and the vector of the second class constraints in the schematic form

$$\begin{aligned} \left\{\phi^a, \Psi^T\right\} &= (0, *, 0, 0, *, *) \,, & \left\{\lambda_a, \Psi^T\right\} &= (*, 0, 0, 0, 0, 0) \,, \\ \left\{h_{ij}, \Psi^T\right\} &= (0, 0, 0, 0, 0, 0) \,, & \left\{\pi^{ij}, \Psi^T\right\} &= (0, 0, 0, 0, *, *) \,. \\ \left\{p_a, \Psi^T\right\} &= (0, 0, 0, 0, *, *) \,. \end{aligned} \tag{2.35}$$

Using this expression and (2.32) we easily find

$$\begin{aligned} \{\phi^a, p_b\}_D &= \{\phi^a, p_b\} = \delta^a_b \,, \\ \{\phi^a(\mathbf{x}), \lambda_b(\mathbf{y})\} &= \triangle^a_b(\mathbf{x}, \mathbf{y}) \,, \\ \left\{\phi^a, \pi^{ij}\right\}_D &= \{\phi^a, h_{ij}\}_D = 0 \,, \\ \left\{h_{ij}, \pi^{kl}\right\}_D &= \left\{h_{ij}, \pi^{kl}\right\} \,, \qquad \left\{\pi^{ij}, \pi^{kl}\right\} = 0 \,, \\ \left\{\pi^{ij}, \lambda_a\right\}_D &= 0 \,. \end{aligned} \tag{2.36}$$

where $\triangle^a_b$ is a non-trivial matrix whose explicit form is not important for us. We see that there is non-trivial phase space structure between ϕ^a and λ_b so that λ_b can be effectively considered as the variable conjugate to ϕ^a. In fact, this follows easily from the structure of the constraints where p_a can be eliminated as follows. First of all $D-1$ momenta $P^b_a p_b$ vanish strongly. On the other hand $p_a\lambda^a$ can be solved using Ω and we obtain

$$\lambda^a p_a = \pm\frac{1}{2}(\lambda^b\lambda_b)\sqrt{h}\sqrt{1+h^{ij}\partial_i\phi\partial_j\phi} \ . \tag{2.37}$$

Further, $\tilde{k}^a, \psi$ vanish strongly so that unrestricted variables are ϕ^a and conjugate veriables λ_a while from Σ we express λ as a function of canonical variables. Inserting these results into the Hamiltonian constraint $\mathcal{H}_T$ we obtain the final result

$$\mathcal{H}_T = \frac{2}{\sqrt{h}}\pi^{ij}\mathcal{G}_{ijkl}\pi^{kl}-\sqrt{h}R\pm p_\phi\sqrt{1+h^{ij}\partial_i\phi\partial_j\phi}+\frac{1}{2}\sqrt{h}\lambda_a h^{ij}\partial_i\phi^a\partial_j\phi+\frac{1}{2}\sqrt{h}V(\phi^a) \ . \tag{2.38}$$

We derived remarkable result that shows that despite of the complexity of the extended model the Hamiltonian constraint still possesses linear dependence on the momentum p_ϕ that is conjugate to the scalar field ϕ as in original mimetic dark energy model. Usually the presence of the linear momentum in the Hamiltonian is sign of an instability. However we can argue that this cannot be the case of the model studied here. To see this in more details let us introduce auxiliary field M with conjugate momentum P_M that is the primary constraint and with the Poisson brackets

$$\{M(\mathbf{x}), P_M(\mathbf{y})\} = \delta(\mathbf{x}-\mathbf{y}) \ . \tag{2.39}$$

With the help of these fields we can rewrite the Hamiltonian constraint $\mathcal{H}_T$ given above to the form

$$\mathcal{H}_T=\frac{2}{\sqrt{h}}\pi^{ij}\mathcal{G}_{ijkl}\pi^{kl}-\sqrt{h}R+\frac{p_\phi^2}{2\sqrt{h}M}+\frac{M}{2}\sqrt{h}(1+h^{ij}\partial_i\phi\partial_j\phi)+\frac{1}{2}\sqrt{h}\lambda_a h^{ij}\partial_i\phi^a\partial_j\phi+\frac{1}{2}\sqrt{h}V(\phi^a) \tag{2.40}$$

which strongly resembles the Hamiltonian constraint for the dust that was carefully analyzed in [13]. In fact solving the equation of motion for M which is equivalent to the requirement of the preservation of the primary constraint $P_M \approx 0$ we obtain $M^2 = \frac{p_\phi^2}{h(1+h^{ij}\partial_i\phi\partial_j\phi)}$. Inserting this result into (2.40) we obtain (2.38). The point is that the scalar field part of the Hamiltonian constraint (2.40) is clearly positive definite on condition that $M>0$ and hence there is no sign of instability. For the case when $M<0$ we can certainly perform trivial canonical transformation $(M, P_M) \to (-M, -P_M)$ that preserves the Poisson bracket (2.39) so that without lost of generality we can presume that $M>0$.

On the other hand there is potentially another source of instability in this theory which is the fact that the theory is linear in λ_a and we argued above that this variable can be considered as the variable conjugate to ϕ^a due to the presence of the non-trivial Dirac brackets between them. We deal with this term in the same way as in the the case of the linear term in p_ϕ. We introduce auxiliary field $N_{ij}=N_{ji}$ with inverse N^{ij} and rewrite the term $\frac{1}{2}\sqrt{\lambda}\lambda_a h^{ij}\partial_i\phi^a\partial_j\phi$ in the Hamiltonian constraint (2.40) as

$$\frac{1}{2}\sqrt{h}\lambda_a h^{ij}\partial_i\phi^a\partial_j\phi \to \frac{1}{4}\sqrt{h}\left(\lambda_a N^{ab}\lambda_b + N_{ab}(h^{ij}\partial_i\phi\partial_j\phi^a)(h^{kl}\partial_k\phi\partial_l\phi^b)\right) \tag{2.41}$$

so that the Hamiltonian constraint (2.40) has the extended form

$$\begin{aligned}\mathcal{H}_T &= \frac{2}{\sqrt{h}}\pi^{ij}\mathcal{G}_{ijkl}\pi^{kl} - \sqrt{h}R + \frac{p_\phi^2}{2\sqrt{h}M} + \frac{M}{2}\sqrt{h}(1 + h^{ij}\partial_i\phi\partial_j\phi)\\ &+\frac{1}{4}\sqrt{h}\left(\lambda_a N^{ab}\lambda_b + N_{ab}(h^{ij}\partial_i\phi\partial_j\phi^a)(h^{kl}\partial_k\phi\partial_l\phi^b)\right) + \frac{1}{2}\sqrt{h}V(\phi^a)\,. \qquad (2.42)\end{aligned}$$

Next we introduce the constraints $P^{ab} \approx 0$, where momenta P^{ab} are conjugate to N_{ab} with following Poisson brackets

$$\left\{N_{ab}(\mathbf{x}), P^{cd}(\mathbf{y})\right\} = \frac{1}{2}\left(\delta_a^c\delta_b^d + \delta_a^d\delta_b^c\right)\delta(\mathbf{x}-\mathbf{y})\,. \qquad (2.43)$$

Then the requirement of the preservation of the constraint $P^{ab} \approx 0$ implies the equation

$$-\lambda_c N^{ca}\lambda_d N^{db} + h^{ij}\partial_i\phi\partial_j\phi^a = 0 \qquad (2.44)$$

that can be solved as $\lambda_c N^{ca} = h^{ij}\partial_i\phi\partial_j\phi^a$. Inserting this result into (2.42) we reproduce the Hamiltonian constraint (2.40). Since we can demand that N^{ab} is positive definitive exactly in the same way as in case of the variable M we find that the scalar field contribution to the Hamiltonian constraint is positive definitive and hence there is no sign of instability which is certainly desired result. We also support this claim with the analysis of simpler model studied in the next section.

3 The case of single scalar field

In this section we focus on much simpler model when the number of additional scalar fields in the action (2.9) is equal to one. In this case it is convenient to introduce a notation $\phi^1 = \psi, \lambda_1 = \omega$ so that

$$p_\phi = \lambda\sqrt{h}\nabla_n\phi + \frac{1}{2}\sqrt{h}\omega\nabla_n\psi\,, \qquad p_\psi = \frac{1}{2}\omega\nabla_n\phi\,, \qquad k_\lambda \approx 0\,, \qquad k_\omega \approx 0 \qquad (3.1)$$

and hence we easily find an inverse transformation

$$\nabla_n\phi = \frac{2}{\sqrt{h}\omega}p_\psi\,, \qquad \nabla_n\psi = \frac{2}{\sqrt{h}\omega}\left(p_\phi - \frac{2\lambda}{\omega}p_\psi\right) \qquad (3.2)$$

and corresponding Hamiltonian

$$\begin{aligned}H &= \int d^3\mathbf{x}(\pi^{ij}\partial_t h_{ij} + p_\phi\partial_t\phi + p_\psi\partial_t\psi - \mathcal{L})\\ &= \int d^3\mathbf{x}(N\mathcal{H}_T + N^i\mathcal{H}_i)\,, \qquad (3.3)\end{aligned}$$

where

$$\begin{aligned}\mathcal{H}_T &= \frac{2}{\sqrt{h}}\pi^{ij}\mathcal{G}_{ijkl}\pi^{kl} - \sqrt{h}R + \frac{2}{\sqrt{h}\omega}p_\psi p_\phi - \frac{2\lambda}{\sqrt{h}\omega^2}p_\psi^2\\ &+\frac{1}{2}\lambda\sqrt{h}(1 + h^{ij}\partial_i\phi\partial_j\phi) + \frac{1}{2}\sqrt{h}\omega h^{ij}\partial_i\psi\partial_j\phi + \frac{1}{2}\sqrt{h}V(\psi)\,,\\ \mathcal{H}_i &= -2h_{ik}D_j\pi^{kj} + p_\phi\partial_i\phi + p_\psi\partial_i\psi\,. \qquad (3.4)\end{aligned}$$

Now requirement of the preservation of the momentum conjugate to λ,ω gives

$$\begin{aligned}\partial_t k_\lambda &= \{k_\lambda, H\} = N\left(\frac{2}{\sqrt{h}\omega^2}p_\psi^2 - \frac{1}{2}\sqrt{h}(1+h^{ij}\partial_i\phi\partial_j\phi)\right) = N\Omega\approx 0\,,\\ \partial_t k_\omega &= \{k_\omega, H\} = N\left(\frac{2}{\sqrt{h}\omega^2}p_\psi p_\phi - \frac{4\lambda}{\sqrt{h}\omega^3}p_\psi^2 - \frac{1}{2}\sqrt{h}h^{ij}\partial_i\psi\partial_j\phi\right) \equiv N\Sigma\approx 0 \end{aligned} \tag{3.5}$$

while the requirement of the preservation of the momenta conjugate to N, N^i again implies two secondary constraints $\mathcal{H}_T\approx 0\,,\mathcal{H}_i\approx 0$. We see that the constraint structure is much simpler than in case of general number of scalar fields ϕ^a. In fact, it is very easy to determine the Poisson brackets between smeared form of the constraints $\mathcal{H}_T\approx 0\,,\mathcal{H}_i\approx 0$

$$\begin{aligned}\{\mathbf{T}_T(N),\mathbf{T}_T(M)\} &= \mathbf{T}_S((N\partial_i M - M\partial_i N)h^{ij})\,,\\ \left\{\mathbf{T}_S(N^i),\mathbf{T}_S(M^i)\right\} &= \mathbf{T}_S((N^i\partial_i M^j - M^i\partial_i N^j))\,,\\ \left\{\mathbf{T}_S(N^i),\mathbf{T}_T(M)\right\} &= \mathbf{T}_T(N^i\partial_i M)\ . \end{aligned} \tag{3.6}$$

Of course, we still have to ensure that $\mathcal{H}_T$ is the first class constraint. This can be easily done using the prescription presented in previous section so that we will not repeat it here but we simply sat that $\mathcal{H}_i\approx 0\,,\tilde{\mathcal{H}}_T\approx 0$ are first class constraints which is a reflection of the full diffeomorphism invariance of the theory.

The number of physical degrees of freedom is obtained via Dirac's formula: (number of canonical variables)/2 — (number of first class constraints) — (number of second class constraints)/2. Using Hamiltonian and spatial diffeomoprhism constraints we can eliminate eight number of degrees of freedom from the gravitational sector with twelfth variables h_{ij},π^{ij} so that we obtain four phase space degrees of freedom corresponding to massless graviton. Further, k_λ, k_ω vanish strongly since they are the second class constraints with Ω and Σ that can be solved for ω and λ. Explicitly, from Ω we find

$$\omega = \pm\frac{2}{\sqrt{h}\sqrt{(1+h^{ij}\partial_i\phi\partial_j\phi)}}p_\psi\,, \tag{3.7}$$

while from $\Sigma=0$ we express λ as

$$\lambda = \frac{\sqrt{h}\omega^3}{4p_\psi^2}\left(\frac{2}{\sqrt{h}\omega^2}p_\psi p_\phi - \frac{1}{2}\sqrt{h}h^{ij}\partial_i\psi\partial_j\phi\right)\ . \tag{3.8}$$

Note that the dependence on λ disappears from the Hamiltonian since the Hamiltonian constraint is linear in λ. Inserting these results into the Hamiltonian constraint we obtain

$$\mathcal{H}_T = \frac{2}{\sqrt{h}}\pi^{ij}\mathcal{G}_{ijkl}\pi^{kl} - \sqrt{h}R \pm\sqrt{1+h^{ij}\partial_i\phi\partial_j\phi}p_\phi \pm\frac{1}{\sqrt{1+h^{ij}\partial_i\phi\partial_j\phi}}p_\psi h^{ij}\partial_i\psi\partial_j\phi + \frac{1}{2}\sqrt{h}V(\psi) \tag{3.9}$$

and we see that the Hamiltonian constraint is linear in p_ϕ and p_ψ. However introducing two auxiliary fields M and K with conjugate momenta $P_M\approx 0$, $P_K\approx 0$ we can rewrite this Hamiltonian constraint into the form[6]

$$\begin{aligned}\mathcal{H}_T = & \frac{2}{\sqrt{h}}\pi^{ij}\mathcal{G}_{ijkl}\pi^{kl} - \sqrt{h}R + \frac{1}{2\sqrt{h}M}p_\phi^2 + \frac{1}{2}\sqrt{h}M(1+h^{ij}\partial_i\phi\partial_j\phi) \\ & + \frac{1}{2\sqrt{h}K}p_\psi^2 + \frac{1}{2}\sqrt{h}K\frac{(h^{ij}\partial_i\phi\partial_j\psi)^2}{1+h^{ij}\partial_i\phi\partial_j\phi} + \frac{1}{2}\sqrt{h}V(\psi) \ .\end{aligned} \tag{3.10}$$

Repeating the same arguments as in previous section we can presume that $M > 0, K > 0$ without lost of generality so that contribution from the scalar field in (3.10) is positive definite and hence it is bounded from below which is satisfactory fact.

Finally we determine schematic form of Dirac brackets between canonical variables. Let us denote the second class constraints as $\Psi_A = (k_\lambda, k_\omega, \Omega, \Sigma)$. Then it ie easy to see that the matrix of the Poisson brackets between second class constraints has the form

$$\Omega_{AB} = \begin{pmatrix} 0 & A \\ -A & B \end{pmatrix} \tag{3.11}$$

with inverse

$$\Omega^{-1} = \begin{pmatrix} A^{-1}BA^{-1} & -A^{-1} \\ A^{-1} & 0 \end{pmatrix} , \tag{3.12}$$

where A, B are 2×2 matrices. Then the Poisson brackets between canonical variables and second class constraints have schematic form

$$\begin{aligned} \left\{h_{ij}, \Psi^T\right\} &= (0,0,0,0) \,, & \left\{\pi^{ij}, \Psi^T\right\} &= (0,0,*,*) \,, \\ \left\{\phi, \Psi^T\right\} &= (0,0,0,*) \,, & \left\{p_\phi, \Psi^T\right\} &= (0,0,*,*) \,, \\ \left\{\psi, \Psi^T\right\} &= (0,0,*,*) \,, & \left\{p_\psi, \Psi^T\right\} &= (0,0,0,*) \ . \end{aligned} \tag{3.13}$$

Then we can easily calculate Dirac bracket for canonical variables. Let us demonstrate this calculation on following examples

$$\begin{aligned} \left\{h_{ij}, \pi^{kl}\right\}_D &= \left\{h_{ij}, \pi^{kl}\right\} - \left\{h_{ij}, \Psi^T\right\}\Omega^{-1}\left\{\Psi, \pi^{kl}\right\} = \left\{h_{ij}, \pi^{kl}\right\} , \\ \left\{\pi^{ij}, \pi^{kl}\right\}_D &= -\left\{\pi^{ij}, \Psi^T\right\}\Omega^{-1}\left\{\Psi, \pi^{kl}\right\} = 0 \end{aligned} \tag{3.14}$$

as follows from (3.12) and (3.13). In the same way we find that all Dirac brackets coincide with corresponding Poisson brackets between canonical variables.

4 $f(\Box\phi)$ Model

In this section we consider Hamiltonian formulation of the model that was proposed in [8, 9]. This model is defined by the action[7]

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left[R(g_{\mu\nu}) - \lambda\left(1 + g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right) + f(\Box\phi)\right] , \tag{4.1}$$

[6]This is possible on condition when $\partial_i\psi \neq 0, \partial_i\phi \neq 0$. Clearly when either ϕ or ψ depend on time only term linear in p_ψ is zero and potential problem with instability disappears.

[7]The special case of this model with $f(\Box\phi) = \gamma(\Box\phi)^2$ was studied recently in [14] where it was introduced as the covariant action for the IR limit of the projectable Hořava-Lifshitz gravity.

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where f is given in (1.8) and $\Box = \frac{1}{\sqrt{-g}}\partial_\mu[\sqrt{-g}g^{\mu\nu}\partial_\nu]$. In order to proceed to the Hamiltonian formalism we introduce two auxiliary fields and rewrite the action into the form

$$\begin{aligned}
S &= \frac{1}{2}\int d^4x\sqrt{-g}\left[R(g_{\mu\nu}) - \lambda\left(1 + g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right) + f(A) + B(A - \Box\phi)\right] \\
&= \frac{1}{2}\int d^4x\sqrt{-g}\left[R(g_{\mu\nu}) - \lambda\left(1 + g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right) + f(A) + BA + \sqrt{-g}\partial_\mu B g^{\mu\nu}\partial_\nu\phi\right] \\
&= \frac{1}{2}\int d^4x\sqrt{-g}\left[R(g_{\mu\nu}) - \lambda\left(1 + g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right) - U(B) + \sqrt{-g}\partial_\mu B g^{\mu\nu}\partial_\nu\phi\right] ,
\end{aligned} \tag{4.2}$$

where in the last step we solved the equation of motion for A that has the form

$$\frac{df(A)}{dA} + B = 0 \tag{4.3}$$

and we presumed solution in the form $A = \Psi(B)\,, f'(\Psi(B)) = -B$. Inserting this result back to the action we obtain the last form of the action with the potential $U(B)$ defined as

$$U(B) = -f(\Psi(B)) - B\Psi(B) \ . \tag{4.4}$$

Note the crucial difference between the action (4.2) and the action studied in previous section which is an absence of the Lagrange multiplier ω. This fact will be very important for the structure of the constraints as we will see below. In order to proceed to the Hamiltonian analysis of this action we write it in $3+1$ formalism

$$\begin{aligned}
S = \frac{1}{2}\int dt d^3\mathbf{x} N\sqrt{h}[&K_{ij}\mathcal{G}^{ijkl}K_{kl} + R + \lambda(-1 + \nabla_n\phi\nabla_n\phi - h^{ij}\partial_i\phi\partial_j\phi) \\
&-\nabla_n B\nabla_n\phi + h^{ij}\partial_i B\partial_j\phi - U(B)] \ .
\end{aligned} \tag{4.5}$$

From (4.5) we easily find

$$\pi^{ij} = \frac{1}{2}\sqrt{h}\mathcal{G}^{ijkl}K_{kl}\,, \qquad p_B = -\frac{1}{2}\sqrt{h}\nabla_n\phi\,, \qquad p_\phi = \sqrt{h}\lambda\nabla_n\phi - \frac{1}{2}\sqrt{h}\nabla_n B \tag{4.6}$$

so that the Hamiltonian has the form

$$H = \int d^3\mathbf{x}(\pi^{ij}\partial_t h_{ij} + p_\phi\partial_t\phi + p_B\partial_t B - \mathcal{L}) = \int d^3\mathbf{x}(N\mathcal{H}_T + N^i\mathcal{H}_i)\,, \tag{4.7}$$

where

$$\begin{aligned}
\mathcal{H}_T &= \frac{2}{\sqrt{h}}\pi^{ij}\mathcal{G}_{ijkl}\pi^{kl} - \frac{1}{2}\sqrt{h}R + \frac{1}{2}\sqrt{h}\lambda(1 + h^{ij}\partial_i\phi\partial_j\phi) \\
&\quad - \frac{2}{\sqrt{h}}p_\phi p_B - \frac{2\lambda}{\sqrt{h}}p_B^2 - \frac{1}{2}\sqrt{h}h^{ij}\partial_i B\partial_j\phi + \frac{1}{2}\sqrt{h}U(B)\,, \\
\mathcal{H}_i &= -2h_{ik}D_j\pi^{jk} + p_\phi\partial_i\phi + p_B\partial_i B \ .
\end{aligned} \tag{4.8}$$

Now the preservation of the primary constraints $\pi_N \approx 0$, $\pi_i \approx 0$ again implies two secondary constraints $\mathcal{H}_T \approx 0$, $\mathcal{H}_i \approx 0$ while the requirement of the preservation of the constraint $p_\lambda \approx 0$ implies

$$\partial_t p_\lambda = \{p_\lambda, H\} = N\left(-\frac{1}{2}\sqrt{h}(1 + h^{ij}\partial_i\phi\partial_j\phi) + \frac{2}{\sqrt{h}}p_B^2\right) \equiv N\Sigma \approx 0 \ . \tag{4.9}$$

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Now we have to require that the constraint $\Sigma(\mathbf{x})$ is preserved during the time evolution of the system. To do this we have to calculate

$$
\begin{aligned}
\partial_t \Sigma &= \left\{ \Sigma, \int d^3\mathbf{y} N \mathcal{H}_T(\mathbf{y}) \right\} \\
& N\left(2\sqrt{h}\partial_i\phi h^{ij}\partial_j\left(\frac{p_B}{\sqrt{h}}\right) - 2\frac{p_B}{\sqrt{h}}\partial_i(h^{ij}\partial_j\phi) + \frac{1}{2}h_{ij}\pi^{ij}(1+h^{kl}\partial_k\phi\partial_l\phi) + \frac{2}{h}h_{ij}\pi^{ij}p_B^2 \right. \\
& \left. +2\partial_i\phi\partial_j\phi\pi^{ij} - \partial_i\phi\partial_j\phi h^{ij}h_{kl}\pi^{kl} - 2p_B\frac{dU}{dB}\right) \\
&\approx N\left(2\sqrt{h}\partial_i\phi h^{ij}\partial_j\left(\frac{p_B}{\sqrt{h}}\right) - 2\frac{p_B}{\sqrt{h}}\partial_i(\sqrt{h}h^{ij}\partial_j\phi) + 2\partial_i\phi\partial_j\phi\pi^{ij} + h_{ij}\pi^{ij} - 2p_B\frac{dU}{dB}\right) \\
&= N\Sigma^{II} ,
\end{aligned}
\tag{4.10}
$$

where in the last step we used constraint Σ. We see that in order to preserve constraint Σ during the time evolution of the system we have to require that either N or Σ^{II} vanish. Clearly the first condition is too strong and it implies singular metric so that it is natural to demand an existence of the new constraint $\Sigma^{II} \approx 0$. In other words we have two second class constraints $\Sigma(\mathbf{x}) \approx 0\,, \Sigma^{II}(\mathbf{x}) \approx 0$. Considering $\mathcal{H}_T, \mathcal{H}_i$ we extend them in the same way as in previous sections and we find that $\tilde{\mathcal{H}}_T, \tilde{\mathcal{H}}_i$ are first class constraints.

Let us now solve the second class constraints Σ, Σ^{II}. The first one can be solved for p_B while the second one can be solved for B if we presume an existence of the inverse function to $\frac{dU}{dB}$. Then we can write $B = B(h_{ij}, \pi^{ij}, p_\phi, \phi)$ and hence the Hamiltonian constraint $\mathcal{H}_T$, after solving the second class constraints, has the form

$$
\begin{aligned}
\mathcal{H}_T &= \frac{2}{\sqrt{h}}\pi^{ij}\mathcal{G}_{ijkl}\pi^{kl} - \frac{1}{2}\sqrt{h}R \pm p_\phi\sqrt{1+h^{ij}\partial_i\phi\partial_j\phi} \\
& -\frac{1}{2}\sqrt{h}h^{ij}\partial_i B(h_{ij}, \pi^{ij}, \phi, p_\phi)\partial_j\phi + \frac{1}{2}\sqrt{h}U(B(h_{ij}, \pi^{ij}, \phi, p_\phi))
\end{aligned}
\tag{4.11}
$$

so that we again find that the Hamiltonian constraint is linear in p_ϕ and we can make it bounded from bellow exactly as in the previous sections. On the other hand we see that there are no additional degrees of freedom as we could expected from the presence of the d'Alembertian $\Box$ in the action. This result can be considered as an confirmation of the claim presented in [8]. Naively we could expect that due to the presence of the higher derivative operator in the action (4.1) Ostrogratsky instability occurs. On the other hand this is strictly true in theory which is non-degenerate while the action (4.1) is degenerate theory which leads to the presence of the constraints in the Hamiltonian formalism that eliminate additional degrees of freedom as we showed above.

Now we briefly mention the form of the Dirac brackets between canonical variables. From the form of second class constraints $\Psi_A = (\Sigma, \Sigma^{II})$ we easily find that it has the form

$$
\Omega = \{\Psi_A, \Psi_B\} = \begin{pmatrix} 0 & A \\ -A & B \end{pmatrix}
\tag{4.12}
$$

so that inverse matrix has the schematic form

$$\Omega^{-1} = \begin{pmatrix} A^{-1}BA^{-1} & -A^{-1} \\ A^{-1} & 0 \end{pmatrix} . \tag{4.13}$$

We again introduce the notation

$$\begin{aligned} \left\{h_{ij}, \Psi^T\right\} &= (0, *) , & \left\{\pi^{ij}, \Psi^T\right\} &= (*, *) , \\ \left\{\phi, \Psi^T\right\} &= (0, 0) , & \left\{p_\phi, \Psi^T\right\} &= (*, *) . \end{aligned} \tag{4.14}$$

Now since π^{ij} and p_ϕ have non-zero Poisson brackets with the primary constraint Σ we obtain that the structure of Dirac brackets is more complicated. For example

$$\left\{h_{ij}, \pi^{kl}\right\}_D = \left\{h_{ij}, \pi^{kl}\right\} + \left\{h_{ij}, \Sigma^{II}\right\} A^{-1} \left\{\Sigma, \pi^{kl}\right\} . \tag{4.15}$$

In the same way we can show that there are non-zero Dirac brackets $\left\{\pi^{ij}, \pi^{kl}\right\}_D$, $\left\{p_\phi, \pi^{ij}\right\}_D$ and so on. In other words the phase space has very complicated structure as opposite to the original form of the mimetic theory.

5 Conclusions

We have studied inhomogeneous mimetic model proposed in [5] from Hamiltonian point of view. We argued that in case of general number of additional scalar fields there are new primary constraints that makes the analysis rather complicated. On the other hand we have shown that despite of this fact the Hamiltonian constraint is linear in momentum p_ϕ conjugate to scalar field ϕ which signals possible instability of this model which is the same situation as in case of the original mimetic model. However we also argue that it is possible to rewrite the scalar part of the Hamiltonian constraint to have the same form as in case of the dust which is well defined system [13] since the Hamiltonian constraint is quadratic and bounded from below.

In the next part of this paper we performed canonical analysis of the model proposed in [8]. We determined structure of the constraints and we again showed that the Hamiltonian constraint is linear in the momentum p_ϕ after solving second class constraints. We also argued that the Dirac brackets on the reduced phase space have non-trivial structure which makes further analysis of this theory rather complicated. On the other hand we mean that it would be very interesting to analyze mimetic theory and its modification following seminal papers [12, 13]. We return to this problem in future.

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